

## Random maps in physical systems

L. TRUJILLO<sup>123(\*)</sup>, J. J. SUÁREZ<sup>3</sup>, AND J. A. GONZÁLEZ<sup>23</sup>

<sup>1</sup> *PMMH (CNRS UMR 7636), ESPCI, 10 rue Vauquelin 75231 Paris Cedex 05 France*

<sup>2</sup> *International Centre for Theoretical Physics (ICTP), Trieste, Italy*

<sup>3</sup> *Centro de Física, IVIC, A.P. 21827, Caracas 1020-A, Venezuela*

PACS. 05.45.-a – Nonlinear dynamics and nonlinear dynamical systems.

PACS. 42.65.Sf – Dynamics of nonlinear optical systems; optical instabilities, optical chaos and complexity, and optical spatio-temporal dynamics.

PACS. 05.45.Vx – Communication using chaos.

**Abstract.** – We show that functions of type  $X_n = P[Z^n]$ , where  $P[t]$  is a periodic function and  $Z$  is a generic real number, can produce sequences such that any string of values  $X_s, X_{s+1}, \dots, X_{s+m}$  is deterministically independent of past and future values. There are no correlations between any values of the sequence. We show that this kind of dynamics can be generated using a recently constructed optical device composed of several Mach-Zehnder interferometers. Quasiperiodic signals can be transformed into random dynamics using nonlinear circuits. We present the results of real experiments with nonlinear circuits that simulate exponential and sine functions.

Recent experiments with electronic circuits have shown the possibility of communication with chaos [1–4]. The interesting question of communication with chaotic lasers has also been discussed in [5]. The fast dynamics displayed by optical systems offers the possibility of communication at bandwidths of hundreds of megahertz or higher. Very recently, Umeno et al. [6] have proposed an optical device implementation of chaotic maps. They rightly claim that the development of secure fiber-optic communication systems can have a large impact on future telecommunications. One problem in this area is constructing all-optical devices for the transmission of high-bit-rate signals with the appropriate security. Umeno et al. introduce multi Mach-Zehnder (MZ) interferometers which implement a very nice class of chaotic maps. Other papers have shown that even chaotic communication systems can be cracked if the chaos is predictable [7, 8].

In the present Letter we will show that using the same experimental setup of Ref. [6] with some small modifications and also other physical systems, it is possible to construct random maps that generate completely unpredictable dynamics.

S. Ulam and J. von Neumann [9, 10] proved that the logistic map  $X_{n+1} = 4X_n(1 - X_n)$  can be solved using the explicit function  $X_n = \sin^2[\theta\pi 2^n]$ . Other chaotic maps are solvable exactly using, e.g., the functions  $X_n = \sin^2[\theta\pi k^n]$ ,  $X_n = \cos[\theta\pi k^n]$ , and other functions of type  $X_n = P[k^n]$ , where  $k$  is an integer [11–14]. For instance,  $X_n = \sin^2[\theta\pi 3^n]$  is the exact

---

(\*) E-mail: leo@pmmh.espci.fr

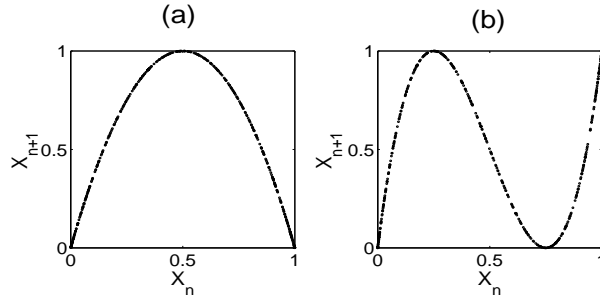


Fig. 1 – First-return maps produced by functions  $X_n = \sin^2[\theta\pi 2^n]$  (a) and  $X_n = \sin^2[\theta\pi 3^n]$  (b).

general solution to the cubic map  $X_{n+1} = X_n(3 - 4X_n)^2$ . The first-return maps for the dynamics of these systems can be observed in Fig. 1.

In Ref. [6] an optical circuit composed of  $N$  MZ interferometers is presented. The scheme of this circuit is shown in Fig. 2. In this experimental setup, the input signal is divided into  $N$  with equal power after passing an  $1 \times N$  coupler. The intensity of light is measured at the output of the  $n$ -th MZ interferometer using a power meter. Each intensity is defined as  $X_n$ . The path length difference is given by  $\Delta L(n)$ . The values of  $\Delta L(n)$  satisfy the relationship

$$\Delta L(n+1) = m\Delta L(n), \quad (1)$$

where  $m$  is an integer,  $m > 1$ .

The transfer function at the output part of the  $n$ -th MZ interferometer is given by the equation  $X_n = \sin^2[\pi\Delta L(n)r/\lambda]$ , where  $\lambda$  is the wavelength of the light source and  $r$  is the effective refractive index of the optical paths of the MZ interferometers. Thus the output powers  $X_1, X_2, \dots, X_N$  satisfy the equation  $X_n = \sin^2[\pi r\Delta L(1)m^n/(\lambda m)]$ .

The authors of Ref. [6] proposed to change the initial conditions by changing the wavelength of the light source. Umeno et al. [6] performed experiments for the cases  $m = 2$  and  $m = 3$ , and  $N = 2$ . They measured the values  $X_1$  and  $X_2$  several times changing the wavelength in the range from 1560.00 nm to 1560.5 nm. The figures they obtained are approximately equivalent to the first-return maps of the logistic and the cubic maps (See Fig. 1.)

We should stress here that any chaotic map of type  $X_{n+1} = f(X_n)$  is predictable in the short term because  $X_{n+1}$  is always defined as a function of the previous value. For instance, for the logistic map, whenever  $X_n \approx 0.1$ ,  $X_{n+1} \approx 0.36$ .

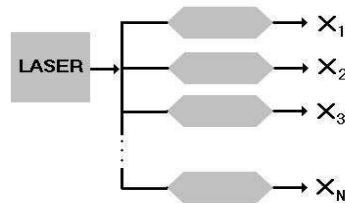


Fig. 2 – Scheme of an experiment described in Ref. [6] with  $N$  Mach-Zehnder interferometers. In this experiment a chaotic sequence is generated.

In the present Letter, we will show that functions of type

$$X_n = P[\theta T Z^n], \quad (2)$$

where  $P[t]$  is a periodic function,  $T$  is the period of  $P[t]$ ,  $\theta$  is a real parameter and  $Z$  is a noninteger number, can generate random dynamics in the sense that  $X_{n+1}$  is not determined by any string  $X_0, X_1, \dots, X_n$  of previous values.

Moreover we will show that the sequence of values  $X_n$  is such that past values cannot be used to predict future values and future values cannot be used to “predict” past values. Furthermore, for irrational  $Z$ , there are no correlations at all between the values of the sequences.

Let us define the family of sequences

$$X_n^{k,m,s} := P \left[ T (\theta_0 + q^m k) \left( \frac{q}{p} \right)^s \left( \frac{p}{q} \right)^n \right], \quad (3)$$

where  $k$ ,  $m$  and  $s$  are integer. The parameters  $k$  distinguishes the different sequences. For all sequences parametrized by  $k$ , the strings of  $m+1$  values  $X_s, X_{s+1}, X_{s+2}, \dots, X_{s+m}$  are the same. This is so because  $X_n^{k,m,s} = P [T \theta_0 (q/p)^s (p/q)^n]$ , for all  $s \leq n \leq m+s$ . So we can have an infinite number of sequences that share the same string of  $m+1$  values. Nevertheless, the next value  $X_{s+m+1}^{k,m,s} = P [T \theta_0 (p/q)^{m+1} + T k p^{m+1}/q]$  is uncertain. In general,  $X_{s+m+1}^{k,m,s}$  can take  $q$  different values. In addition, the value  $X_{s-1}$ , ( $X_{s-1} = P [T \theta_0 (q/p) + T k q^{m+1}/p]$ ), is also undetermined from the values of the string  $X_s, X_{s+1}, X_{s+2}, \dots, X_{s+m}$ . There can be  $p$  different possible values for  $X_{s-1}$ . Thus, for any string  $X_s, X_{s+1}, X_{s+2}, \dots, X_{s+m}$ , the future and the past are both uncertain. In the case of a generic irrational  $Z$ , there are infinite possibilities for the future and the past.

We should remark that the functions (2), with a noninteger  $Z = p/q$ , are not in the class  $X_{n+1} = f(X_n)$  nor  $X_{n+1} = f(X_n, \dots, X_{n-m+1})$ , however they can be expressed as random maps of type  $X_{n+1} = f(X_n, I_n)$ , where  $I_n$  is a time-dependent random variable (See Refs. [15, 16]). That is, the fact that  $Z$  is not integer leads to a sort of time dependence.

Some properties of function  $X_n = \sin^2[\theta \pi Z^n]$ , which is a particular case of (2), have been already studied in previous papers (See e.g. [15–19].)

Here we will show that there are no statistical correlations between  $X_n$  and  $X_m$  (where  $n \neq m$ ). We will investigate the functions  $U_n = \cos(\theta \pi Z^n)$ , which possess zero mean. Note that  $X_n = 1 - U_n^2$ . The values of  $U_n$  are found in the interval  $-1 \leq U_n \leq 1$ . Let us define the  $r$ -order correlations [20, 21]:

$$E(U_{n_1} U_{n_2} \cdots U_{n_r}) := \int_{-1}^1 dU_0 [\rho(U_0) U_{n_1} U_{n_2} \cdots U_{n_r}]. \quad (4)$$

The invariant density  $\rho(U)$  is given by  $\rho(U) = 1/(\pi \sqrt{1 - U^2})$  and  $U_0 = \cos(\pi \theta)$ . We have the following formula for the correlation functions  $E(U_{n_1} U_{n_2} \cdots U_{n_r}) = \int_0^1 d\rho [\cos(\theta \pi Z^{n_1}) \cos(\theta \pi Z^{n_2}) \cdots \cos(\theta \pi Z^{n_r})]$ . Considering that  $\cos \theta = \frac{1}{2} (e^{i\theta} + e^{-i\theta})$ , we obtain

$$E(U_{n_1} U_{n_2} \cdots U_{n_r}) = 2^{-r} \sum_{\sigma} \delta(\sigma_1 Z^{n_1} + \sigma_2 Z^{n_2} + \cdots + \sigma_r Z^{n_r}, 0), \quad (5)$$

where  $\sum_{\sigma}$  is the summation over all possible configurations  $(\sigma_1, \sigma_2, \dots, \sigma_r)$ , with  $\sigma = \pm 1$ , and  $\delta(n, m) = 1$ , if  $n = m$  or  $\delta(n, m) = 0$ , if  $n \neq m$ . We will have non-zero correlations only for the sets  $(n_1, n_2, \dots, n_r)$  that satisfy the equation

$$\sum_{i=1}^r \sigma_i Z^{n_i} = 0, \quad (6)$$

where  $\sigma_i = \pm 1$ .

It is easy to see that, for  $Z > 1$ ,

$$E(U_n U_m) = 0 \quad (7)$$

if  $n \neq m$ . As a particular case  $E(U_n U_{n+1}) = 0$ . We also wish to show that the correlation functions  $E(U_n^i U_{n+1}^j)$  are zero when  $i$  is even and  $j$  is odd (or vice versa) i.e.

$$E(U_n^i U_{n+1}^{2j+1}) = 0. \quad (8)$$

So let us calculate  $E(U_n^i U_{n+1}^{2j+1})$ :  $E(U_n^i U_{n+1}^{2j+1}) = \frac{1}{2^\pi} \sum_{\sigma} \delta(\sigma_1 Z^n + \dots + \sigma_i Z^n + \sigma_{i+1} Z^{n+1} + \dots + \sigma_{i+2j+1} Z^{n+1}, 0)$ . Note that  $E(U_n^i U_{n+1}^{2j+1})$  is not zero only if  $(\sigma_1 + \sigma_2 + \dots + \sigma_i) Z^n + (\sigma_{i+1} + \sigma_{i+2} + \dots + \sigma_{i+2j+1}) Z^{n+1} = 0$ . For irrational  $Z$ , this equation has not solutions.

Now we wish to consider all the possible correlations  $E(U_{n_1} U_{n_2} \dots U_{n_r})$ . We should note that  $E$  can be non-zero in some “trivial” cases (for instance  $E(U_n^{2j})$ ,  $E(U_n^{2j} U_{n+1}^{2j})$ ) which are related to moments  $E(U_n^{2j})$ . This does not affect randomness [20, 21]. In the language of equations  $\sum_{i=1}^{\sigma} \sigma_i Z^{n_i} = 0$ , this can happen only due to trivial cases as the following  $Z^n - Z^n - Z^{n+1} + Z^{n+1} = 0$ .

We will show that all the “nontrivial” correlations are zero for our functions. Suppose  $n_r = 2j + 1$ . Then  $E(U_{n_1} U_{n_2} \dots U_{n_r})$  is not zero only if there are solutions for the equations  $\sum_{i=1}^r \sigma_i Z^{n_i} = 0$ . But these equations can be written in the form

$$N_0 + N_1 Z + \dots + N_{2j+1} Z^{2j+1} = 0, \quad (9)$$

where  $N_i$  are integer, and  $N_{2j+1} \neq 0$ . For transcendent irrational  $Z$  this equation is never satisfied. Thus the sequences generated by function  $X_n = \sin^2[\theta \pi Z^n]$  with a transcendent  $Z$  are completely uncorrelated.

Different aspects of the predictability problem are used in references [22–25] as a way to characterize complexity and to find distinction between noise and chaos in experimental time series.

In Refs. [22–25] several quantities are introduced in order to determine the true character of the time series. All these methods have in common that one has to choose certain length scale  $\epsilon$  and a particular embedding dimension  $m$ . The mentioned quantities discussed in these articles display different behaviors as the resolution is varied. According to these different behaviors one can distinguish chaotic and stochastic dynamics.

Using the results of Refs. [15–19] and the present paper, it is possible to prove that functions (2) can represent different kinds of dynamics: chaotic time series (with integer  $Z$ ), random maps or unpredictable sequences (with noninteger  $Z$ ), and completely uncorrelated sequences of independent values (with generic irrational  $Z$ .)

Our functions can be investigated analytically and their complexity can be calculated exactly using theoretical considerations [15, 16]. So these functions can be used as very suitable models in order to check the predictions of Refs. [22–25].

Moreover, we can produce long sequences of values using our models and then, we can study them as experimental time series in the framework of the methods presented in Refs. [22–25]. In fact, we have investigated the asymptotic behavior of the quantities discussed in Ref. [22–25], and our results coincide with those obtained in the mentioned papers. Additionally, we have checked the formula  $K = \lambda \theta(\lambda) + h$  for the complexity of random maps of type  $X_{n+1} = f(X_n, I_n)$ , where  $K$  is the complexity of the system,  $\lambda$  is the Lyapunov exponent of the map and  $h$  is the complexity of  $I_n$ , and  $\theta(\lambda)$  is the Heaviside step function [24]. Details

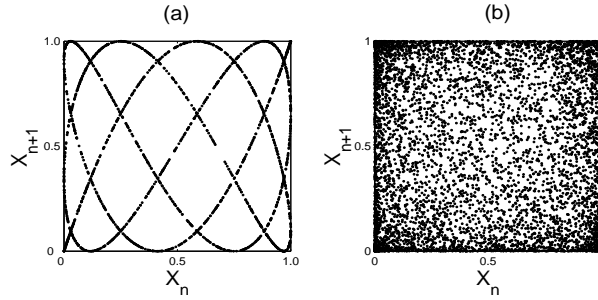


Fig. 3 – First-return maps produced by function  $X_n = \sin^2[\theta\pi Z^n]$  with  $Z = 1.8$  (a) and  $Z = \pi$  (b).

of the applications of our results in the problem of distinguishing a chaotic system from one with intrinsic randomness will be presented in a more extended paper.

The statistical properties of pseudo-random number generators are discussed in Refs. [26]. The authors of these papers have noticed that almost all pseudo-random number generators calculate a new pseudo-random number  $X_{n+1}$  using a recursive formula that depends on the preceeding values  $X_{n+1} = f(X_n, X_{n-1}, \dots, X_{n-r+1})$ . They have found that the failure of these generators in different simulations can be attributed to the low entropy of the production rule  $f()$  conditioned on the statistics of the input values  $X_n, X_{n-1}, \dots, X_{n-r+1}$ . Besides, all these generators have very strong correlations even at the macrostate level used in the simulations [26].

We agree with these researchers that this approach, based on the properties of the generator rule, is more profound than the empirical tests.

In this same spirit, we should say that the rule (2) produces a dynamics where the future values are not determined by the past values. In fact, they can be completely uncorrelated.

Now suppose that we have the same experimental setup of Ref. [6], which is represented in Fig. 2 schematically, but the equation for the path length differences  $\Delta L(n)$  will satisfy a relationship very similar to Eq. (1), that is  $\Delta L(n+1) = Z\Delta L(n)$ , with the change that  $Z$  is not an integer. In this case the sequence  $X_n$  of measured light intensities will be unpredictable.

Fig. 3 shows different examples of the dynamics that can be produced by function  $X_n = \sin^2[\theta\pi Z^n]$  with different noninteger  $Z$ .

Using the properties of function (2) and further investigation we can obtain the following results. Function  $X_n = P[\phi(n)]$ , where  $P[t]$  is a periodic function and  $\phi(n)$  is a non-periodic oscillating function with intermittent intervals of truncated exponential behavior, would produce also unpredictable dynamics.

Furthermore, we can construct functions of type  $X_n = h[\phi(n)]$ , again with very complex behavior, where  $h(t)$  is a non-invertible function and  $\phi(n)$  is, as before, a non-periodic oscillating function with intermittent intervals of truncated exponential behavior. Some chaotic systems can produce the kind of behavior needed for  $\phi(n)$ . This physical system can be constructed, for example, with circuits: a chaotic circuit and a circuit with a non-invertible  $I$ - $V$  characteristic [27]. An experiment with this scheme is reported in [28].

However the most interesting fact is that we can construct function  $\phi(n)$  without using previously produced chaotic signals. In this case we plan to use as input to the non-linear system only regular signals.

A consequence of our theory is that a time-series constructed using three periodic signals can be transformed into an unpredictable dynamics. The theoretical result is that the following

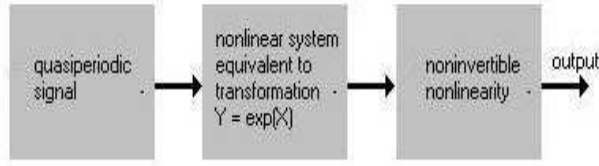


Fig. 4 – Experimental setup to produce unpredictable dynamics using a quasiperiodic signal.

function can be unpredictable:

$$X_n = P[A \exp[Q(n)]], \quad (10)$$

where  $P[y]$  is a periodic function (in some cases it can be just a non-invertible function,) and  $Q(n)$  is a quasiperiodic function represented by the sum of several periodic functions.

As an illustrating example let us study the following function

$$X_n = \sin[\phi(n)], \quad (11)$$

where  $\phi(n) = A \exp[Q(n)]$ ,  $Q(n) = P_1(n) + P_2(n) + P_3(n)$ ,  $P_i(n) = a(n - kT_i)$ , when  $kT_i \leq n \leq (k+1)T_i$ . Here  $T_2/T_1$ ,  $T_3/T_2$  and  $T_3/T_1$  are irrational numbers.

Note that the functions  $P_i(n)$  are piece-wise linear. Function  $Q(n)$  is also piece-wise linear, but it is not periodic. On the other hand, function  $\phi(n)$  will behave as a non-periodic oscillating function with intermittent intervals of finite exponential behavior. At these intervals, function  $X_n$  behaves as function  $X_n = \sin^2[\theta\pi Z^n]$ .

Similar properties can be found in function (11) if  $Q(n) = a_1 \sin(\omega_1 n) + a_2 \sin(\omega_2 n) + a_3 \sin(\omega_3 n)$ . In fact, functions  $\sin(\omega_i n)$  behaves approximately as increasing linear functions whenever  $\omega_n \approx 2\pi k$  where  $k$  is an integer. We have performed real experiments using the setup represented in Fig. 4.

In our experiments, a quasiperiodic time-series was used as input to an electronic circuit that simulates an exponential function [29]. The output of the exponential system is taken as the input to a non-linear system that simulates the sine-function [30]. Fig. 5 shows an example of the dynamics produced by the experiment. Details of the experiment will be presented elsewhere in a more extended paper.

In conclusion, we have shown that functions of type  $X_n = P[\theta TZ^n]$ , where  $P[t]$  is a  $T$ -periodic function,  $\theta$  and  $Z$  are real numbers, can generate random dynamics in the sense that

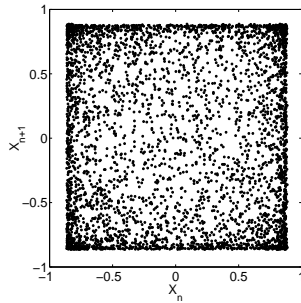


Fig. 5 – Dynamics produced by the experiment.

any string of values  $X_s, X_{s+1}, \dots, X_{s+m}$  is deterministically independent of past and future values. Furthermore, there are no correlations whatsoever between the values of the sequence.

The experimental setup schematically represented in Fig. 2 (See Ref. [6]), where the path-length differences in the Mach–Zehnder interferometers satisfy the equation  $\Delta L(n+1) = Z\Delta L(n)$  (with noninteger  $Z$ ), can be used to produce unpredictable dynamics.

We have performed real experiments with systems that are equivalent to the scheme represented in Fig. 4. These experiments corroborate our prediction that, using just static non-linear systems, a quasiperiodic signal can be transformed into a random signal.

## REFERENCES

- [1] PECORA L. M. AND CARROL T. L., *Phys. Rev. Lett.*, **64** (1990) 821
- [2] PECORA L. M. AND CARROL T. L., *Phys. Rev. A*, **44** (1991) 2371
- [3] DITTO W. L. AND PECORA L. M., *Sci. Am.*, **269** (1993) 62
- [4] CUOMO K. M. AND OPPENHEIM A. V., *Phys. Rev. Lett.*, **71** (1993) 65
- [5] VANWINGGEREN G. D. AND ROY R., *Science*, **279** (1998) 1198
- [6] UMENO K., AWAJI Y. AND KITAYAMA K., in DING. M., DITTO W. L., PECORA L. M., SPANO M. L. (Editor), *Proceedings of the fifth experimental chaos conference* (World Scientific, London) 2001
- [7] CHEN Y. Y., *Europhys. Lett.*, **34** (1996) 24
- [8] PÉREZ G. AND CERDEIRA H., *Phys. Rev. Lett.*, **74** (1995) 1970
- [9] ULAM S. M. AND VON NEUMANN J., *Bull. Am. Math. Soc.*, **53** (1947) 1120
- [10] STEIN P. AND ULAM S. M., *Rosprawy Matematyczne*, **39** (1964) 401
- [11] ADLER R. L. AND RIVLIN T. J., *Proc. Am. Math. Soc.*, **15** (1964) 794
- [12] GEISEL T. AND FAIREN V., *Phys. Lett. A*, **105** (1984) 263
- [13] KATSURA S. AND FUKUDA W., *Physica A*, **130** (1985) 597
- [14] UMENO K., *Phys. Rev. E*, **55** (1997) 5280
- [15] GONZÁLEZ J. A., REYES L. I. AND GUERRERO L. E., *Chaos*, **11** (2001) 1
- [16] GONZÁLEZ J. A., TRUJILLO L. AND ESCALANTE A., *Physica A*, **324** (2003) 723
- [17] GONZÁLEZ J. A. AND CARVALHO L. B., *Mod. Phys. Lett. B*, **11** (1997) 521
- [18] GONZÁLEZ J. A., MARTÍN-LANDROVE M. AND TRUJILLO L., *Int. J. Bifurcation and Chaos*, **10** (2000) 1867
- [19] GONZÁLEZ J. A. AND REYES L. I., *Europhys. Lett.*, **55** (2001) 610
- [20] BECK C., *Nonlinearity*, **4** (1991) 1131
- [21] HILGERS A. AND BECK C., *Physica D*, **156** (2001) 1
- [22] SUGIHARA G. AND MAY R., *Nature*, **344** (1990) 734; WALES D. J., *ibid.*, **350** (1991) 485; TSONIS A. A. AND ELSNER J. B., *ibid.*, **358** (1992) 217
- [23] CENCINI M., FALCIONI M., OLBRICH E., KANTZ H. AND VULPIANI A., *Phys. Rev. E*, **62** (2000) 427
- [24] BOFFETTA G., CENCINI M., FALCIONI M. AND VULPIANI A., *Phys. Rep.*, **356** (2002) 367
- [25] KANTZ H. AND OLBRICH E., *Physica A*, **280** (2000) 34
- [26] BAUKE H. ET AL., *J. Stat. Phys.*, **114** (2004) 1149; MERTENS S. ET AL., *Preprint*, (2003) cond-mat/0305319
- [27] CHUA L. O., DESOER C. A. AND KU E. S., *Linear and nonlinear circuits* (McGraw–Hill, New York) 1987
- [28] GONZÁLEZ J. A., REYES L. I., SUÁREZ J. J., GURRERO L. E. AND GUTIÉRREZ G., *Physica A*, **316** (2002) 259
- [29] SHEIAGOLD D. G., *Nonlinear circuits handbook* (Analog devices, Norwood, Massachusetts, 1976) 1976
- [30] MAGERLEIN J. H., *Rev. Sci. Instrum.*, **49** (1978) 486